UFOs – UNIDENTIFIED FIGURATIVE OBJECTS
A GEOMETRIC CHALLENGE

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The title, though possibly provocative, points at an identification task in algebraic geometry that is far from being trivial or resolved: the *Geometric Recognition Problem*. It consists in detecting from given geometric objects – in this case varieties – their algebraic definition. More precisely, algebraic descriptions of objects which have the same visual properties as the given ones (there may not be uniqueness).

The article presents views of 24 real algebraic surfaces of degree mostly less than 7. The challenge is to find suitable equations, respectively parametrizations, for all of them. Of course, the pictures only allow $\mathcal{C}^1$-interpretations, since the human eye cannot recognize the failure of $\mathcal{C}^2$-differentiability. To indicate geometrically higher order features one had to depict the (iterated) Nash-modifications or transforms of the varieties under blowups.

Each picture comes with a short description of its main geometric configurations which may help to reveal the genesis. Already here we have to be somewhat vague – there seems to be no generally accepted formal language to describe geometry. We do not propose here any such language, nor a procedure how to find algebraic equations for the surfaces. This is a vast field and could be the subject of further investigations. In this sense the article shall only serve as an appetizer: First, to contemplate geometric forms in order to find their exact mathematically rigorous description. Secondly, to search for algebraic procedures in order to construct surfaces from such a description using basic building blocks (e.g., by moving curves or by deforming toric surfaces).

We do not indicate appropriate equations and parametrizations for the surfaces which appear in this article. This is on purpose. Otherwise, the reader could quickly verify that the algebraic description is ok and pass to the next one. No! It is much more puzzling to try yourself finding an equation of an object which you believe to capture geometrically perfectly well.

The Geometric Recognition Problem may initiate new ideas and techniques to algebraic geometry, and not only over the reals or in dimension two. It may thus represent an (even though minor) counterbalance to the domination of algebraic, analytic and topological methods in geometry.

To satisfy the curiosity of the interested reader (not everybody will have time to determine the algebraic origin of the surfaces) we publish daily in december 2005 the equation or parametrization of *one* surface of the article in form of a calendar. The information can be found in the net at `www.hh.hauser.cc`. There, also a short animation of the surface (rotation, deformation, zooming, ...) is shown.

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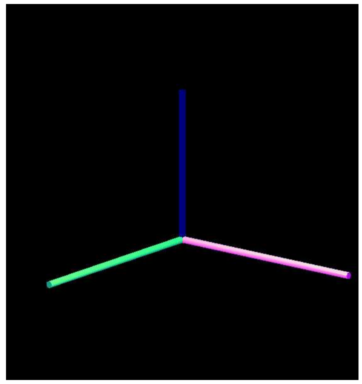
Before passing to the series of pictures, we briefly describe the rules of the game. The coordinate system is in general fixed as follows: x red, y green, z blue.



This may, however, vary according to the position of the camera. The camera always looks at the origin – in a few cases with slight deviations if required to have an authentic view. In the figures, the surfaces are clipped either with a ball or a box (mostly centered at the origin).

By *component* we shall always mean *visual component* (in a heuristic sense) with respect to real three-space and not in the algebraic sense as an irreducible component with respect to the Zariski topology. The same remark applies to the singular locus (the algebraic one and the one we see).

As mentioned before, the pictures can only provide $\mathcal{C}^1$-information. This is a drawback which produces a certain ambiguity when searching for equations. We therefore indicate occasionally additional properties which can help to focus the search on specific ranges of equations.

The pictures were produced by the authors with the ray-tracing program POV-Ray. One way to find the equations or parametrizations would be to scan the pictures, to choose a sufficiently narrow grid and to interpolate from the resulting reference points. This is not what we have in mind or what we are interested in. Our approach is synthetic: Find the algebraic description of the surface by understanding algebraically its geometric properties and construction rules and by then expressing them accurately in a suitable language.

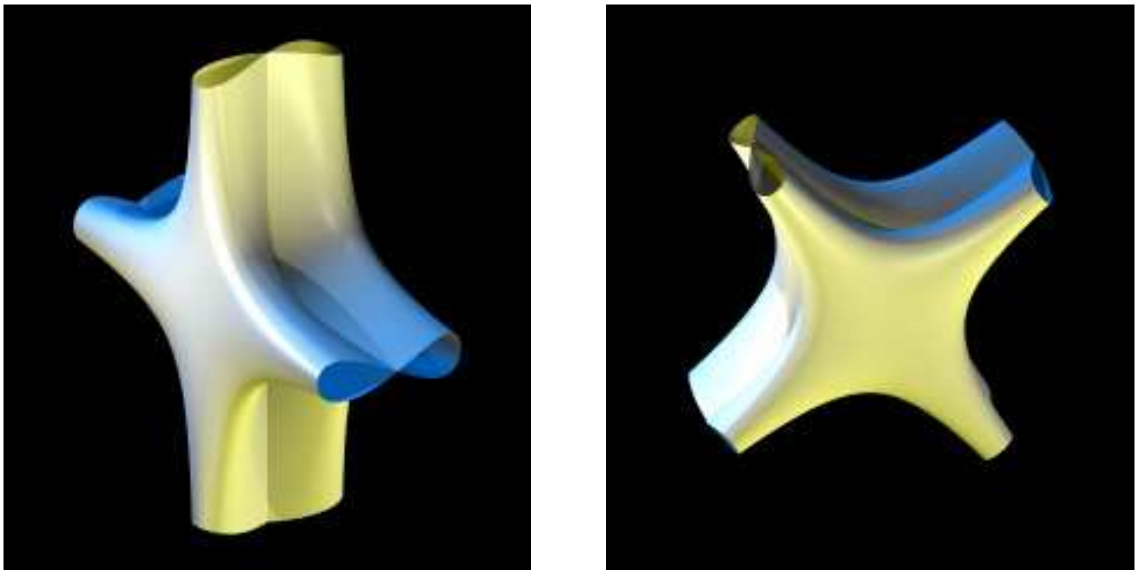
Achtung: There are no results nor theorems in this paper, not even the description of a construction or an algorithm. This can be seen as an *affront* in comparison to the traditional way of publishing in mathematics. It could be! Nevertheless, the authors believe that mathematical research is based on observing and understanding interesting phenomena. The investigation is often more valuable (and satisfactory) than the result itself. Here, with this article, we wish to start such an investigation, not to conclude it.

It is amazing to see how mathematicians, when confronted with visualizations of surfaces they prove theorems about, are sometimes surprised to realize that the geometric object is indeed the subject of their investigation. We made this experience at several occasions. It shows that geometry and geometric contemplation have lost importance – despite the fact that algebraic geometers try to describe and understand geometric phenomena.

The names of the surfaces were chosen by the authors. The reader will have no problems in finding the appropriate translation to her/his language.

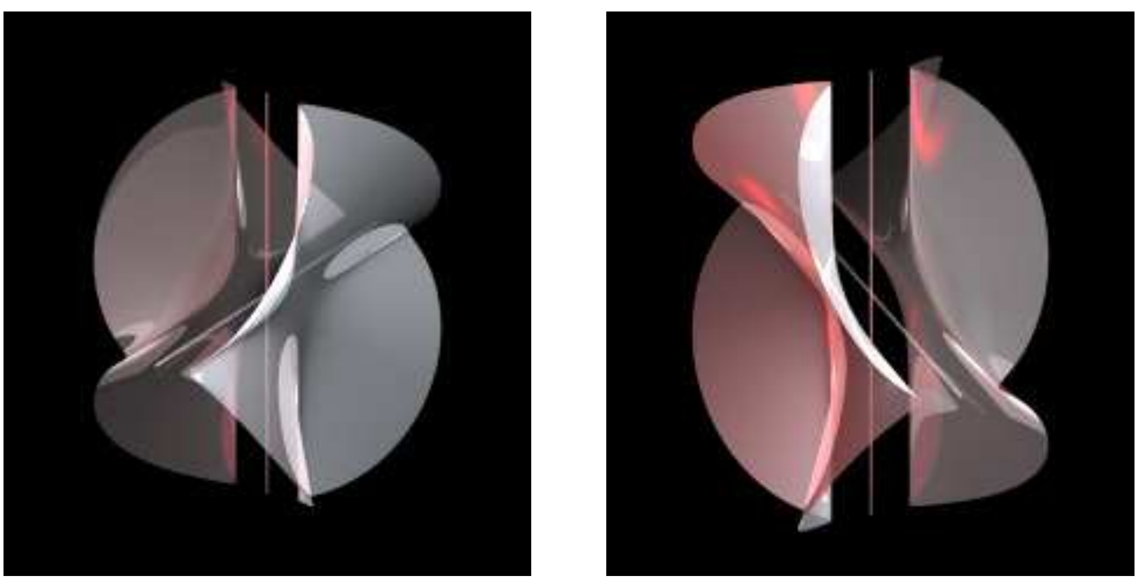
1 **Helix:** degree 4

We start with a surface whose cross-sections with planes parallel (but different) to the coordinate planes look like lemniscates (for $x = c$ and $z = c$) or unions of two hyperbolas (for $y = c$). The section with $y = 0$ is the union of the x- and z-axis. The surface has reflections about the xz-plane and 90° rotations around the y-axis as finite symmetries. More generally, one may ask for helices with an arbitrary number of petals.



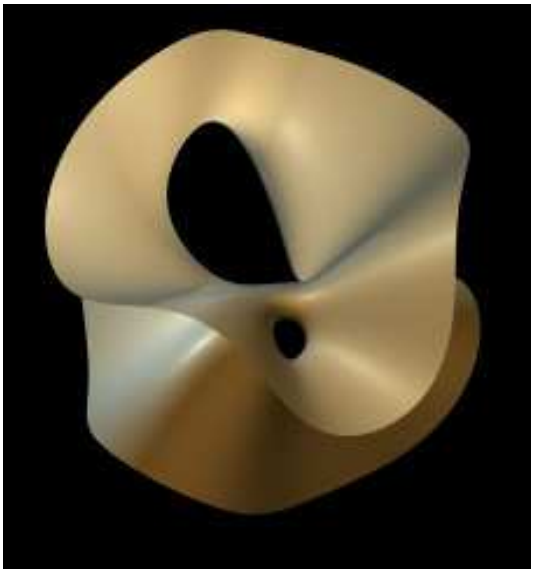
2 **Tanz:** degree 4

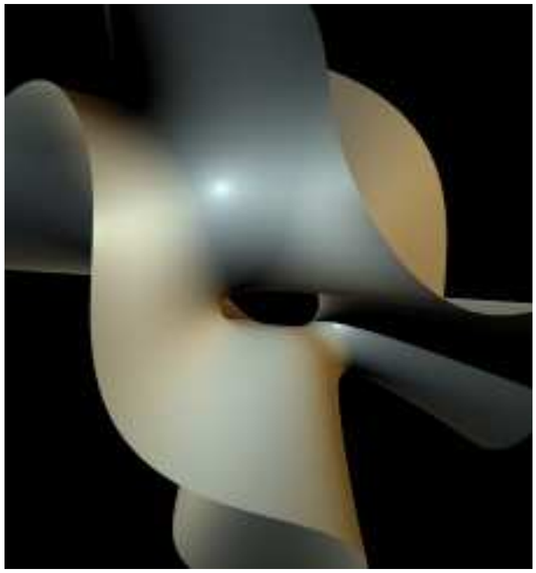
In this surface, two two-dimensional (visual) components are opposed to each other by a reflection about the xz-plane. In addition, the x- and the z-axes form one-dimensional components. The intersections with the planes $x = \pm 1$ are translates of the y- and z-axis. The complexification of the surface coincides with the complexification of the Helix.



3 Columpius: degree 4

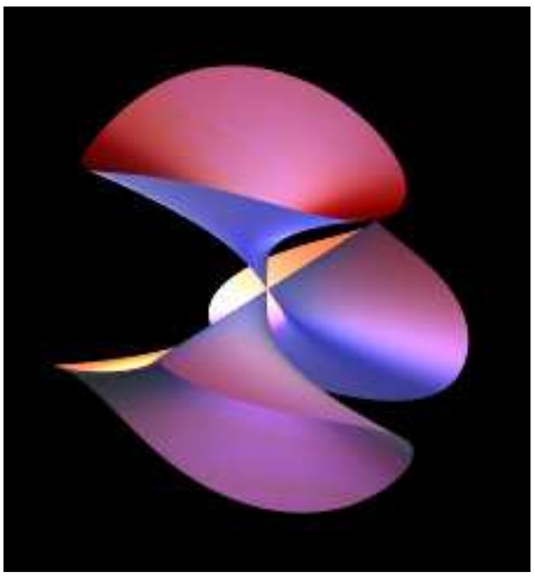
Here, the original equation is difficult to find from inspection. In the picture, it is a perturbation of the equation $x^3y + y^3z + z^3x = 0$ of the Klein Quartic producing three holes. The third hole is "perpendicular" to the two others and appears in the second figure. To relax the problem, one can ask for finding the equation of surfaces with a prescribed configuration of holes.

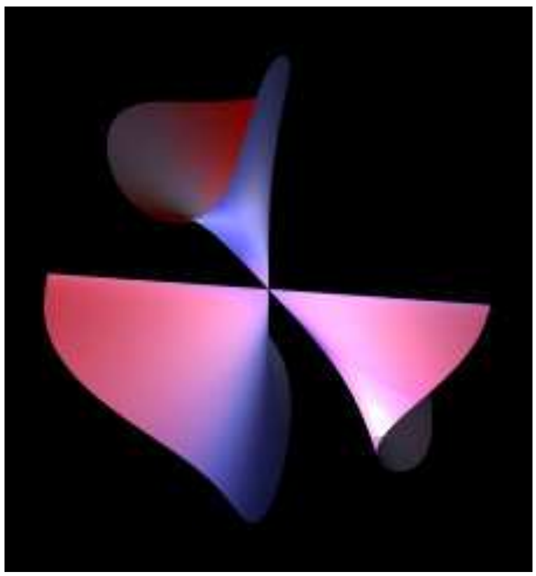




4 Schneeflocke: degree 5

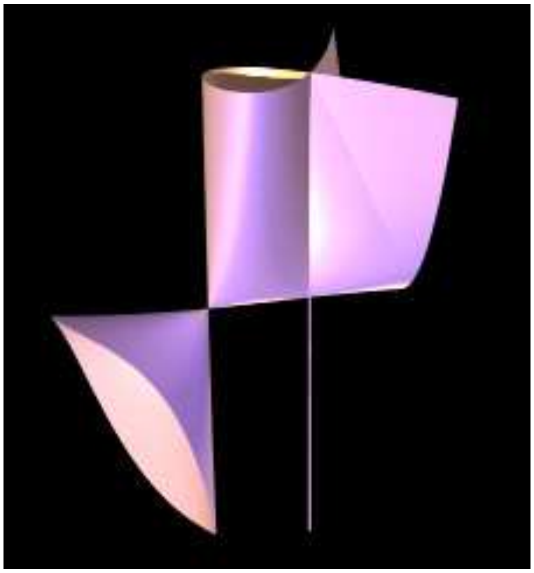
The surface is formed by three components meeting at the origin. Two of them have the y-axis as singular locus and are generated by a contracting tacnode. The third component lies above the z-plane and has a funnel like shape. The surface contains the y- and z-axis and the diagonal $x = y + z = 0$.

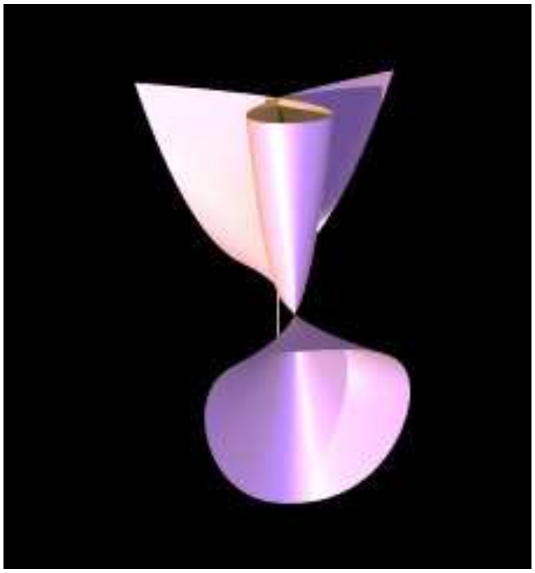




5 **Pirat:** degree 4

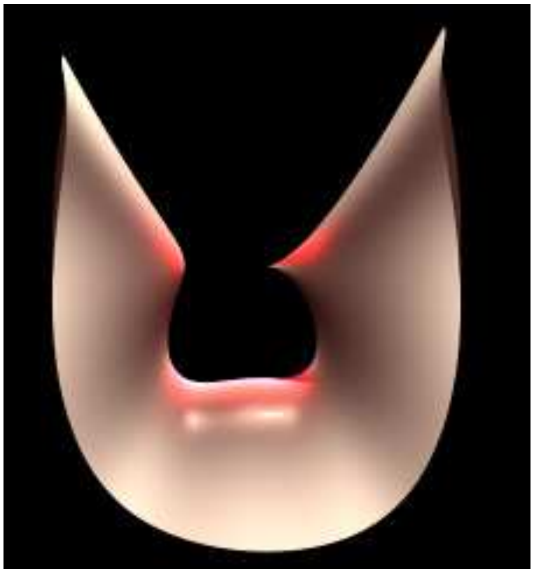
This is the surface one obtains when blowing up the the origin of the Whitney umbrella $x^2 = y^2z$. At the meeting point of the one-dimensional component with the surface we have the local singularity of the Whitney umbrella (a vertical singular axis). The isolated singularity of the surface is an ordinary double point.

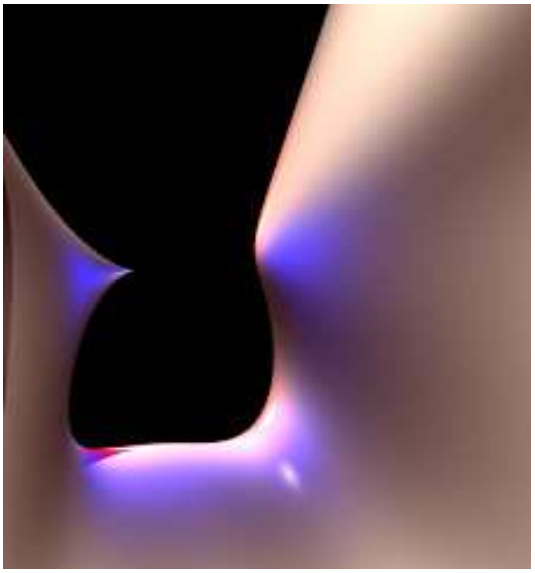




6 **Vis-à-vis:** degree 4

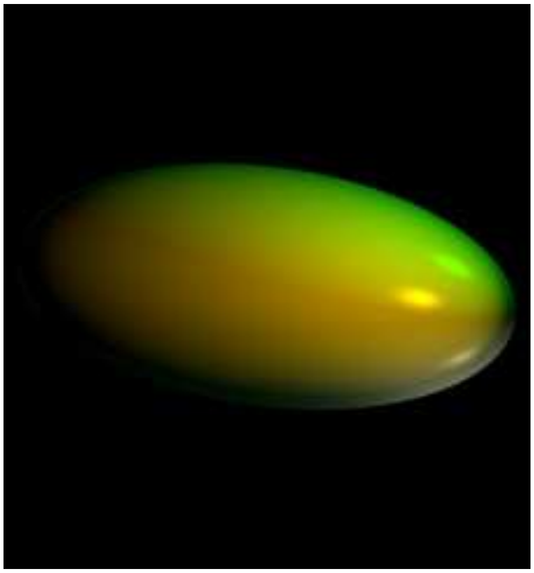
Here, a cusp like isolated singularity is opposed to a local maximum (when considered in the direction of the horizontal axis). This surface is a perturbation of the surface Flirt further on.

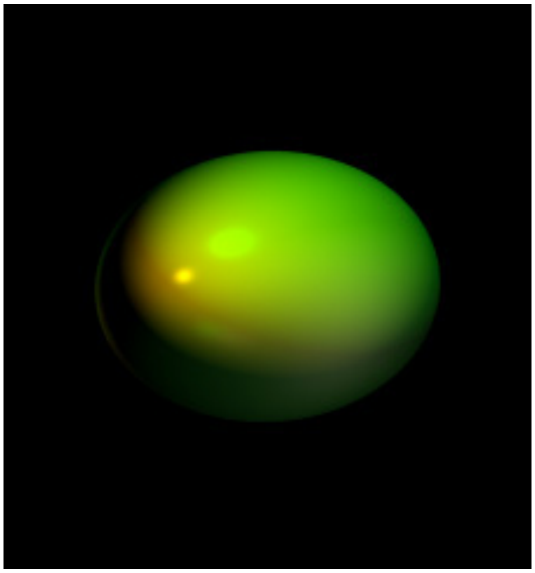




7 **Dattel:** degree 2

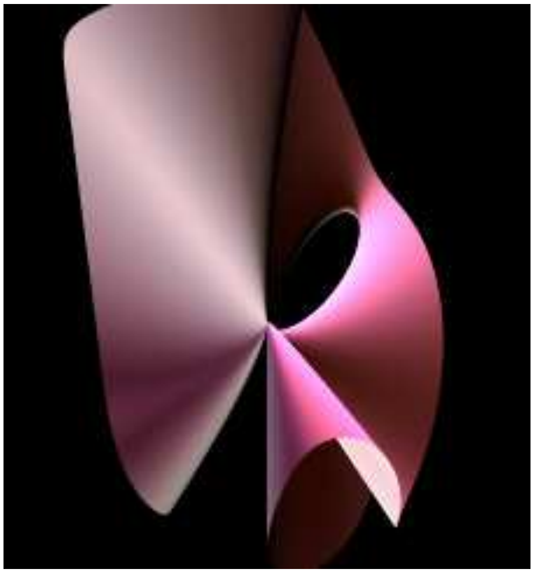
The equation of the ellipsoid is well known.

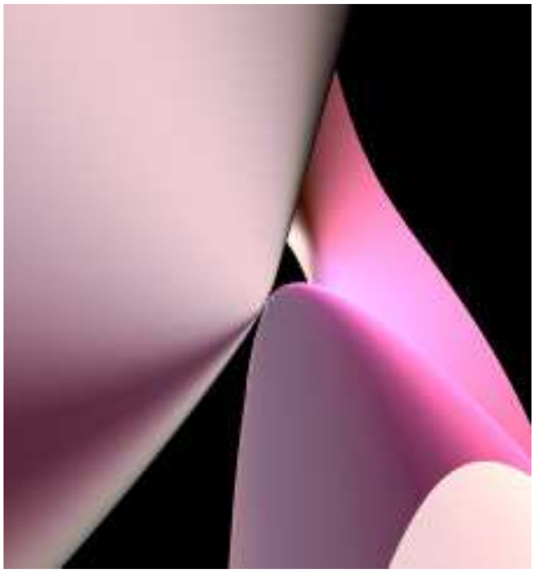




8 **Nadelöhr:** degree 6

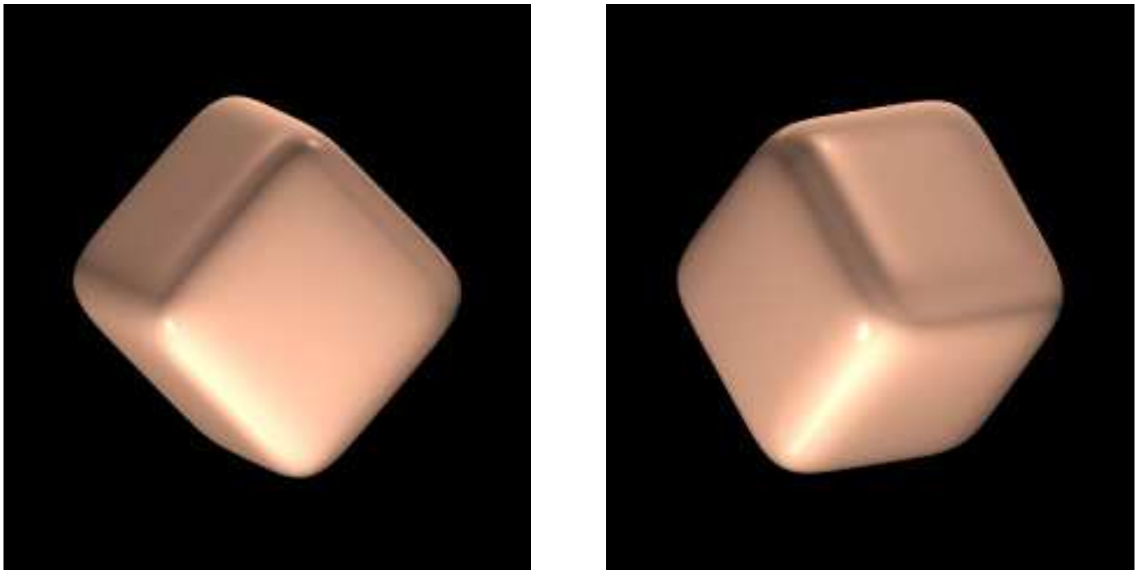
In contrast to the preceding, a suitable equation for the surface is hard to find. The main feature, aside the hole, is the self-tangency at the origin. Moreove, the right hand part has an isolated singular point there, created by the vanishing of a moving parabola.





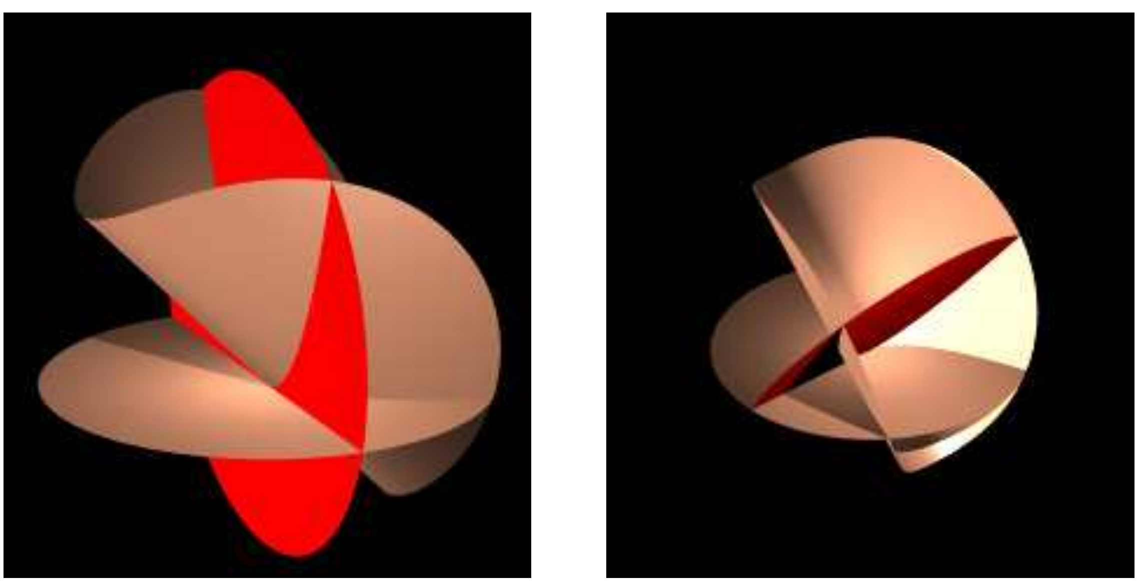
9 Cube: degree 4 or 6

The equation of the rounded cube is relatively easy to determine, due to its symmetry properties. Taking an equation of degree 6 instead of 4 improves the approxiamtion with an exact cube.



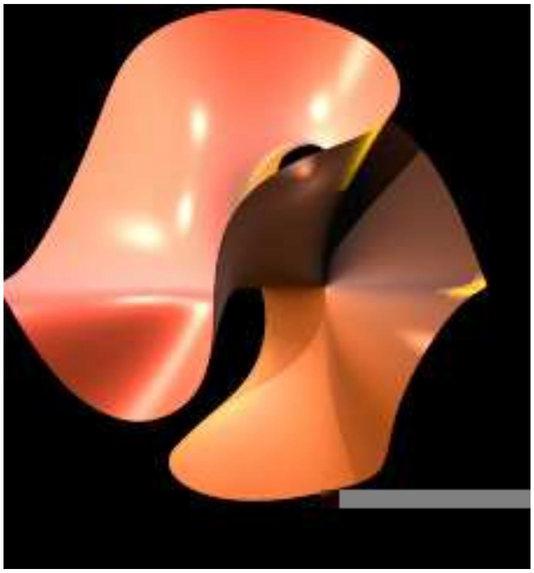
10 Tülle: degree 4

Two planes and one smooth surface intersect pairwise along the x-axis, respectively two parabolas, all three tangent to each other. This is an example of a non-normal crossings singularity with three components, where all pairwise intersections are smooth, but the common intersection is (scheme-theoretically) singular. In the second figure, the vertical plane has been tilted.



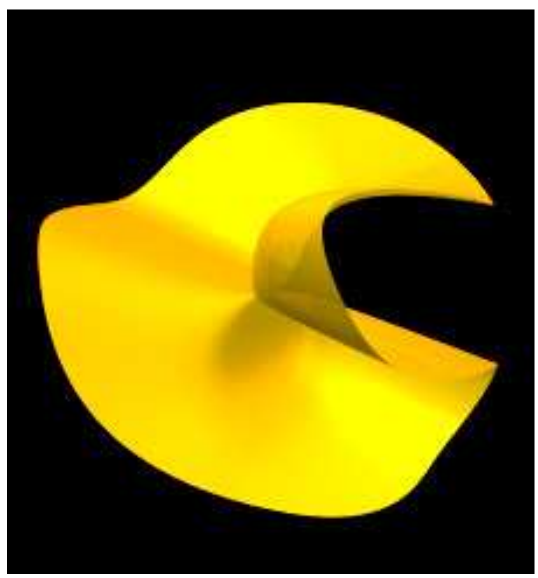
11 **Durchblick:** degree 4

Another deformation of the Klein Quartic, but with just one hole (cf. with Columpius). The ondulation of the intersection of the surface with the sphere is typical. The cross-section of the hole is close to a tacnode (closed curve with one cusp singularity).



12 **Lilie:** degree 5

This Monet-like surface is hard to describe with words. The singularity at 0 seems to be isolated (in the real picture), but the algebraic singular locus is more complicated.



13 Süss: degree 6

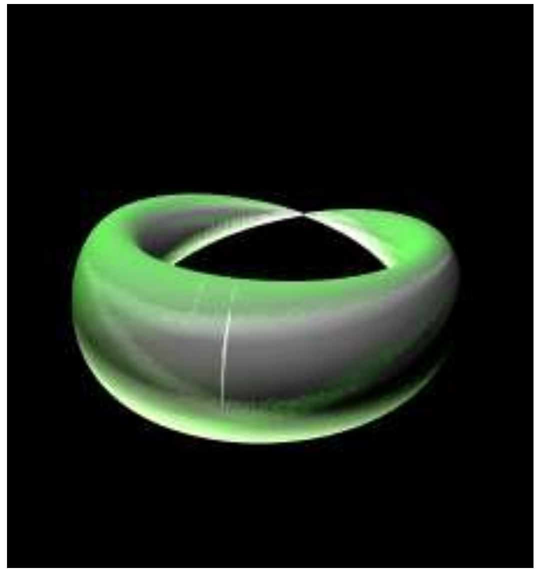
The equation of the heart can easily be found in the net.

14 Quaste: degree 9

The cartesian product of the plane cusp with the node gives a surface in $\mathbb{R}^4$. Here, we see an embedding into 3-space by taking a suitable projection. The singular locus is the union of a cusp and a node, meeting transversally at the origin. The parametrization is much simpler than the equation (the latter involves 24 monomials).

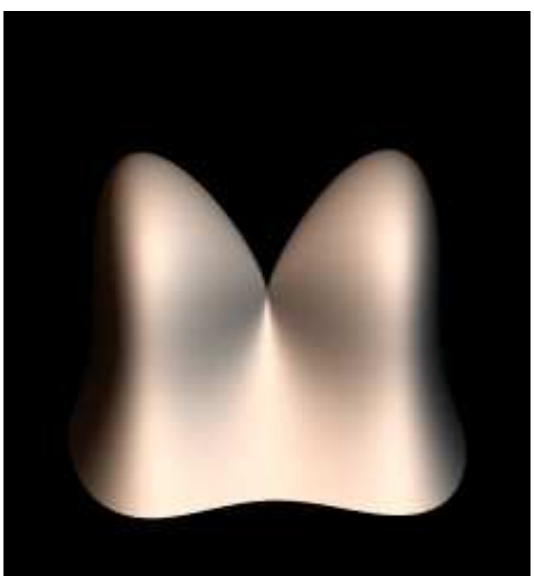
15 Croissant:

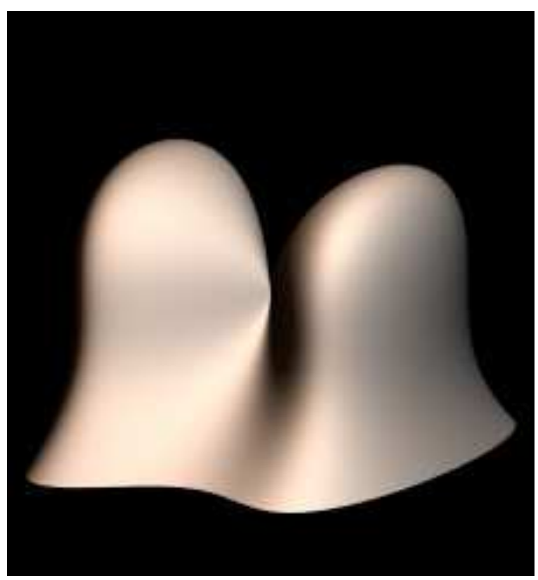
The pinched torus has a complicated equation, so that it is easier to determine a parametrization. The isolated singularity is cone-like (i.e., an ordinary double point). This surface is a typical example for illustrating characteristic classes and radial vector fields.



16 Dromo: degree 4

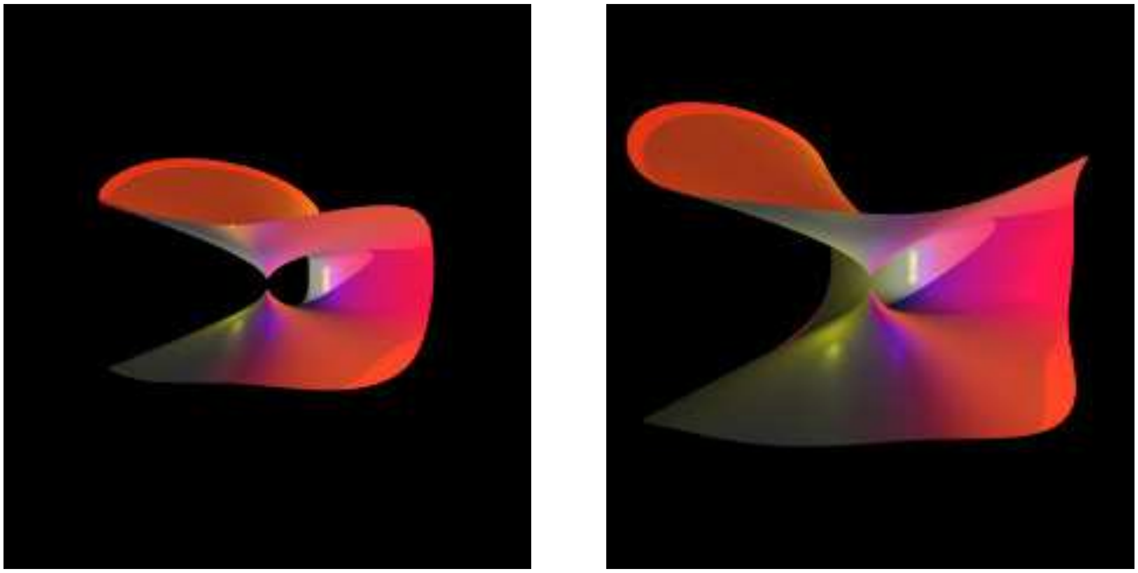
The symmetry of this surface is given by reflection about the yz-plane. The singularity at 0 is similar to the one of the lilie (though simpler). The cross-section with the plane $y = 0$ gives a plane curve which is a variation of the ordinary cusp $x^2 = z^3$.





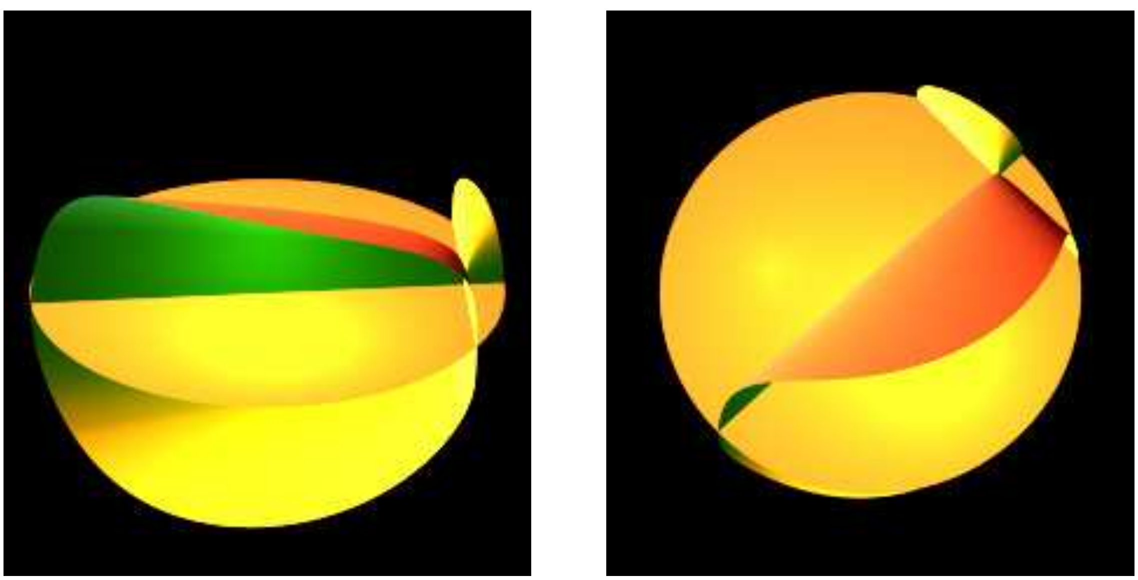
17 **Flirt:** degree 4

This surface is a deformation of Vis-à-vis. Two cusp like singularities meet with the same tangent at 0. Globally, they form one hole, whose cross-section with the plane $x = 0$ is (essentially) a circle.



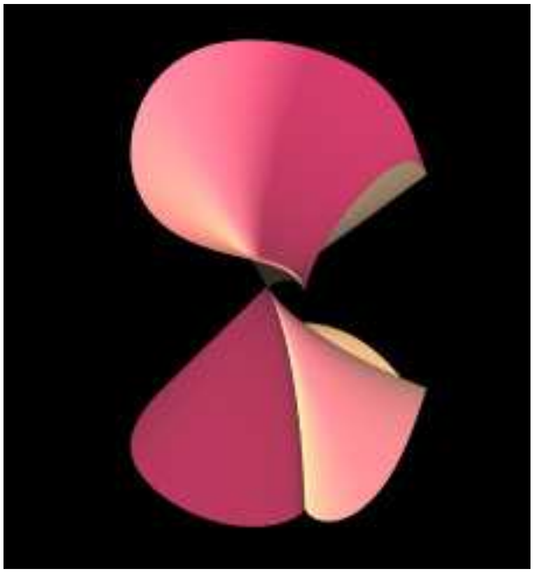
18 **Zeppelin:** degree 5

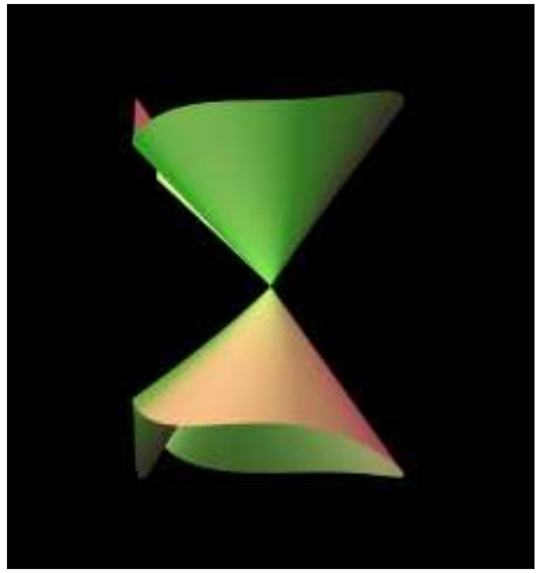
Along the y-axis, we have outside the origin a normal crossings singularity given by two transversally intersecting planes. At the origin, one of the components winds around (with tangent plane equal to the xz-plane), the other continues straight ahead.



19 Daisy: degree 6

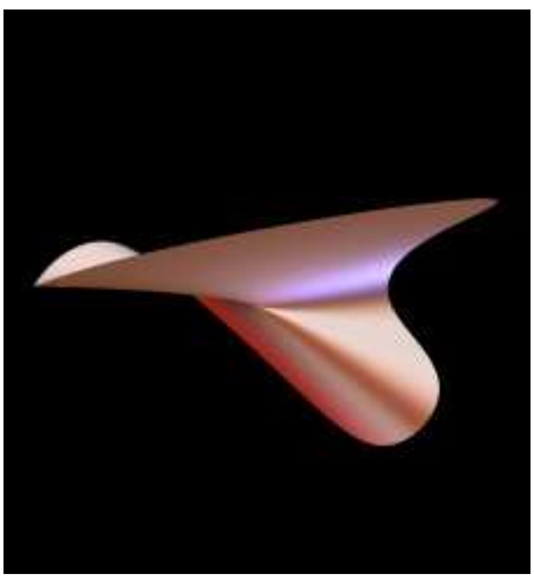
The singular locus consists of two components, two cusps, meeting transversally at the origin. The two components of the surface have cone like shape taken over a closed curve with two cusp singularities (the intersection of the surface with a sphere).

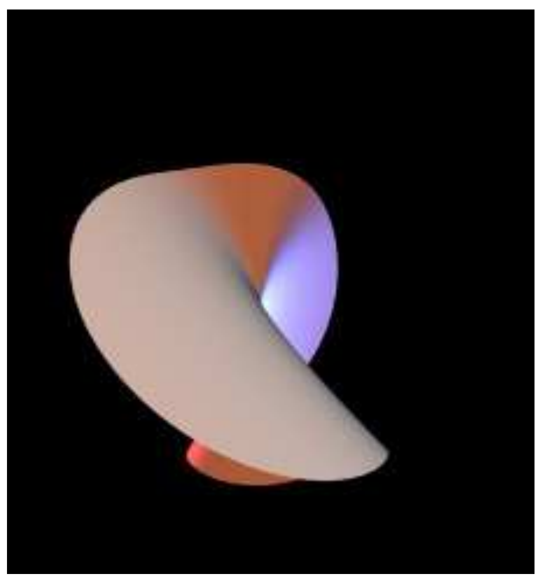




20 Clip: degree 5

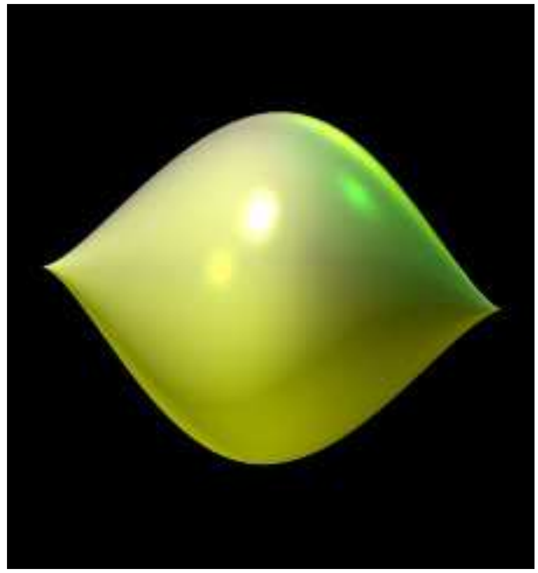
Despite its modest appearance, the surface has an interesting algebraic background. Its (algebraic) singular locus is the twisted space curve with parametrization (t^3, t^4, t^5). The equation of our example has minimal degree with this property. The singular curve cannot be seen, because outside the origin the surface is $\mathcal{C}^1$-smooth along its (algebraic) singularities.





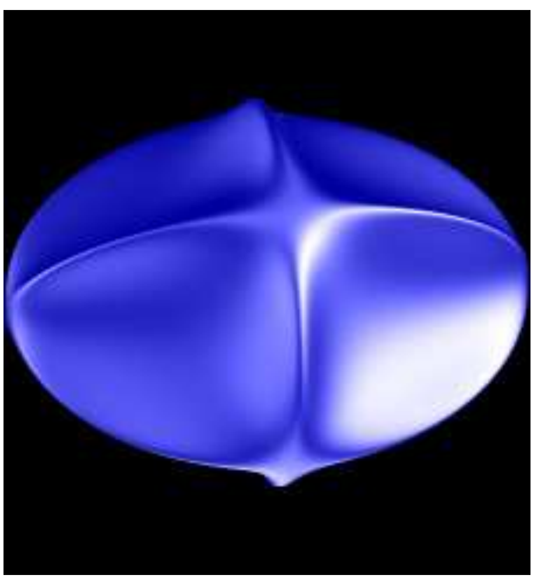
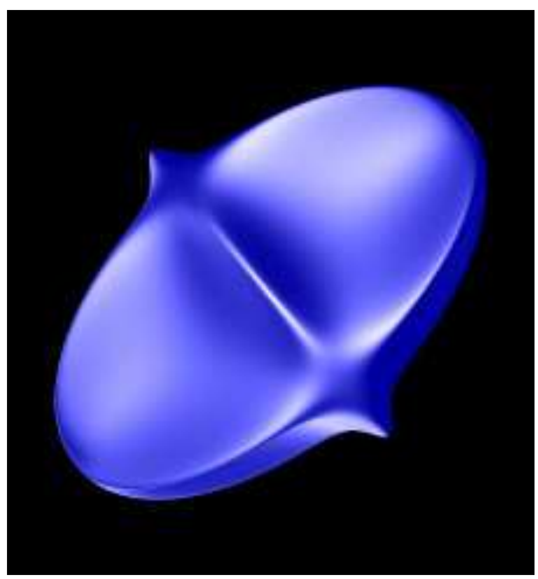
21 Zitrus: degree 6

Surface of revolution of a plane curve with two cusp like singularities obtained by rotation about the y-axis. There is a reflection symmetry with respect to the xz-plane.

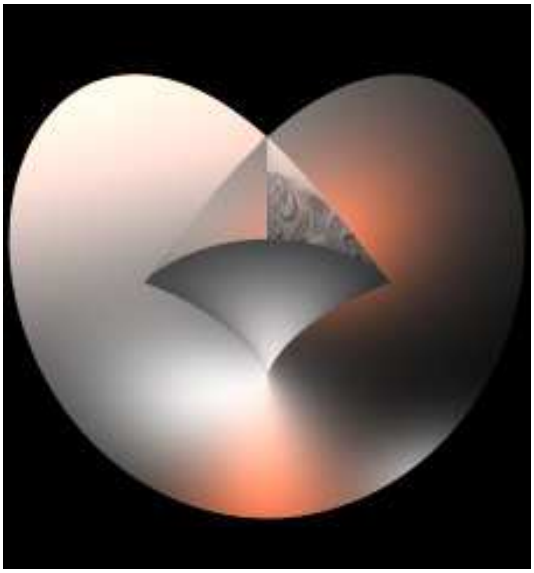
22 Leopold: degree 6

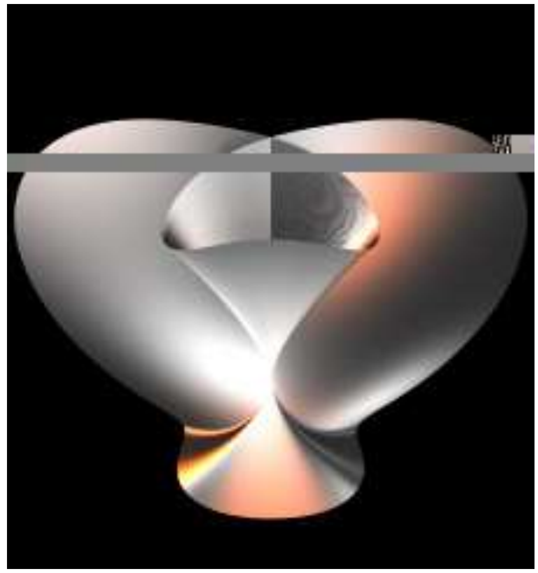
The ellipsoid can be deformed by squeezing it towards the origin outside the coordinate planes. This already gives a good hint how to find the equation of the present surface. It is symmetric with respect to the permutation of x, y and z (up to a homothety).

23 **Taube:** degree 5

Famous example of singularity theory, the discriminant of the miniversal deformation of an A_3-singularity. The value of the coefficients play a decisive role. The second figure is obtained by changing one coefficient of the defining equation.





24 **Distel:** degree 6

Similar construction as for Leopold, stretching a sphere towards the origin except at its six intersection points with the coordinate axes. In the figure, the perturbation coefficient was chosen rather big so as to produce the acute points.